Lecture notes on Dynamic Meteorology prepared by Somenath Dutta for old advanced course.

Chapter-VI

GLOBAL ANGULAR MOMENTUM BUDGET

Some useful concepts:

Momentum: It is the motion due to combined effect of mass (m) and velocity ($\vec{v}$). It is a vector quantity given by $m\vec{v}$. For unit mass momentum is $\vec{v}$.

Angular momentum: It is a vector quantity defined by moment of momentum. The word 'moment' arises in case of rotation only.

Thus if an object of mass (m) and velocity ($\vec{v}$) rotates about a fixed point or about an axis (called axis of rotation), then its angular momentum is given by $\vec{r} \times m\vec{v}$, where, $\vec{r}$ is the position vector of the rotating object. It is a vector quantity.

Now consider a stationary object placed on a circular ring rotating with angular velocity $\vec{\Omega}$ about an axis of rotation. Then, its linear velocity is $\vec{\Omega} \times \vec{r}$

In such case, angular momentum of the object will be $\vec{r} \times (\vec{\Omega} \times \vec{r}) = \vec{\Omega}|\vec{r}|^2$.

Newtons Law:

From the second law of motion we know that

$\frac{d(m\vec{v})}{dt} = \vec{F}$, $\vec{F}$ being the applied force.

So, $\vec{r} \times \frac{d(m\vec{v})}{dt} = \vec{r} \times \vec{F}$

Or, $\frac{d(\vec{r} \times m\vec{v})}{dt} = \vec{r} \times \vec{F}$. LHS is the rate change of angular momentum and RHS is the torque applied by the force $\vec{F}$. Whenever a force $\vec{F}$ is applied to a body, then a tendency of rotation about an axis is generated in the body. This tendency of rotation of the body about that axis is called the torque applied by that force $\vec{F}$ about that axis.

Thus Newton's law states that rate of change of angular momentum about an axis is equal to the torque applied by the forces. Thus if the vector sum of torque is zero, then angular momentum remains conserved. This is known as conservation of angular momentum.

Governing equation for global angular momentum budget:

We consider an unit mass at latitude φ, with zonal velocity u. Then its absolute zonal angular momentum about the earths axis of rotation is $(\Omega + u\, a\cos\varphi)a\cos\phi =$ M (say). To find out an expression for global angular momentum, we consider an infinitesimal volume, $d\sigma$ with density ρ of the atmosphere. So, the angular momentum of this infinitesimal volume is $\rho M\, d\sigma$. Thus the angular momentum of the entire global atmosphere is given by, $A = \iiint_\sigma \rho M\, d\sigma$.

Then the governing equation for global angular momentum budget is given by

$$\frac{\partial A}{\partial t} = -\int_0^\infty \int_{\phi_0}^{\pi/2} \int_0^{2\pi} \vec{\nabla}.(\rho M\vec{V})\, d\lambda\, d\phi\, dz - \int_{\phi_0}^{\pi/2} \int_0^\infty r\Delta p\, dz\, d\phi + \int_0^\infty \int_{\phi_0}^{\pi/2} \int_0^{2\pi} \rho\, rF\, d\lambda\, d\phi\, dz$$

The first term is known as the meridional transport of angular momentum, which signifies the mechanism of transporting zonal angular momentum in the meridional direction (N – S).

The second term, which arises due to E-W pressure difference along a latitude circle, is known as mountain torque term. It is named so, because pressure difference 'Δp' is mainly due to the difference of pressure between windward and leeward side of a section of mountain along that latitude circle.

The third term is known as frictional torque term, and is due to the torque produced by the frictional force.

Discussion about different terms:

Meridional transport of angular momentum: It can be shown that, this term is

$$= \frac{2\pi a\cos\phi}{g}\int_{P_s}^{0} \bar{u}\,\bar{v}\, dp + \frac{2\pi\Omega a^2\cos^2\phi}{g}\int_{P_s}^{0} \bar{v}\, dp + \frac{2\pi a\cos\phi}{g}\int_{P_s}^{0} \overline{u'v'}\, dp$$, where,

$\frac{2\pi a\cos\phi}{g}\int_{P_s}^{0} \bar{u}\,\bar{v}\, dp$ is called the *drifting term*,

$\frac{2\pi\Omega a^2\cos^2\phi}{g}\int_{P_s}^{0} \bar{v}\, dp$ is called the *omega transport term* and

$\frac{2\pi a\cos\phi}{g}\int_{P_s}^{0} \overline{u'v'}\, dp$ is called the *eddy transport term*.

The drifting term signifies the meridional transport of mean zonal angular momentum across the latitude circle ϕ_0 by the mean meridional circulation.

The omega transport term signifies the meridional transport of zonal angular momentum, possessed solely due to earths rotation ($\Omega a \cos\phi$), by the mean meridional circulation.

The eddy transport term signifies the meridional transport of eddy zonal momentum by eddies. For north east-south west oriented westerly troughs, there is less equator ward transport of eddy zonal angular momentum to its rear and more pole ward transport of eddy zonal angular momentum ahead of it. So there is a net pole ward transport of eddy zonal angular momentum for such oriented westerly troughs. Thus NE-SW oriented westerly troughs transports eddy zonal angular momentum.

Mountain torque term: As already been mentioned Δp is due to the difference of pressure between the windward and lee ward side of a mountain barrier.

It is known that, based on the direction of prevailing zonal component, the entire global atmosphere may be categorized into two regimes, viz., the westerly regime and easterly regime.

In the westerly regimes wind ward side is to the west of the barrier and the leeward side is to the east of the barrier. Similarly, in the easterly regimes wind ward side is to the east of the barrier and the leeward side is to the west of the barrier.

Since Δp is measured as, $\Delta p = P_{East} - P_{West}$, hence as shown in the adjoining figure, in the westerly regimes, $\Delta p < 0$. Now as the second term is accompanied with a minus (-) sign, hence the contribution of this term is positive in the westerly regimes and it is negative in the easterly regimes.

Hence the presence of mountain enhances westerly angular momentum in the westerly regimes and it reduces that in the easterly regimes.

Frictional torque term: In the easterly regime, frictional force reduces the strength of easterly wind, that in turn reduces easterly momentum and easterly angular momentum. This is equivalent to say that in the easterly regime, friction increases the westerly angular momentum. Hence, there is a net gain in the westerly angular momentum in the easterly regimes, due to friction. Following the similar argument it can be said that there is a net loss in the westerly angular momentum in the westerly regimes, due to friction.